Generalized Inverse Theorem for Schoenberg Operator

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***Abstract*-For Schoenberg splines of degree $k \ge 1$ -fixed and $n \to \infty$, we establish inverse result in terms of the second order classical moduli. As a consequence, we generalize the earlier inverse estimates of Berens-Lorenz $(n=1)$ and of Beutel, Gonska etc. for (*k* = 1, 2, 3).**

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I. MAIN RESULT

We start with the definition of variation-diminishing operator, introduced by I. Schoenberg. For the case of equidistant knots, we denote it by $S_{n,k}$. Consider the knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k}, n \ge 1, k \ge 1$ with equidistant "interior knots", namely

$$\Delta_n : x_{-k} = \ldots = x_0 = 0 < x_1 < x_2 < \ldots < x_n = \ldots = x_{n+k} = 1$$

and $x_i = \dfrac{i}{n}$ for $0 \le i \le n$. For a bounded real-valued function *f* defined over the interval [0,1] the variation-diminishing spline operator of degree *k* w.r.t. Δ_n is given by

$$S_{n,k}(f,x) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x) \qquad (1.1)$$

for $0 \le x < 1$ and

$$S_{n,k}(f,1) = \lim_{y \to 1, y < 1} S_{n,k}(f,y)$$

with the nodes (Greville abscissas)

$$\xi_{j,k} := \frac{x_{j+1} + \ldots + x_{j+k}}{k}, \; -k \le j \le n-1 \qquad (1.2)$$

and the normalized B-splines as fundamental functions

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, x_{j+1}, \ldots, x_{j+k+1}](. - x)_+^k .$$

The classical second order modulus is defined by

$$\omega_2(f;\delta) := \sup_{0<h\le\delta} \sup_{x \pm h \in [0,1]} |f(x-h) - 2f(x) + f(x+h)| .$$

If, for $x \in [0,1]$, we define $\varphi(x) := \sqrt{x(1-x)}$, then the second order Ditzian–Totik modulus is given by

$$\omega_2^{\varphi}(f;\delta) := = \sup_{0<h\le\delta} \sup_{x \pm h\varphi(x) \in [0,1]} |f(x - h\varphi(x)) - 2f(x) + f(x + h\varphi(x))| .$$

In the case $(n=1, k \ge 1)$ Schoenberg operator reduces to the classical Bernstein operator and in this case the following inverse result was proved by Berens and Lorentz in 1972 (see, for example, [4], p. 313).

Theorem A. *Let* $0 < \alpha < 2$. *There exists a constant* $c = c(\alpha) > 0$ *such that whenever* $f \in C[0,1]$ *and*

$$|f(x) - B_k f(x)| \le M . \left(\frac{x(1-x)}{k} \right)^{\frac{\alpha}{2}},$$

$x \in [0,1]$, $k = 1,2.,\ldots,$

then $\omega_2(f;h) \le c.M.h^{\alpha}$, $h > 0$.

The corresponding direct result was obtained earlier by J.-d. Cao -[3]. Later Ditzian -[5], D.- x. Zhou -[11], Felten -[6] generalized this:

Theorem B. *Let* $\varphi(x) = \sqrt{x(1-x)}$, $\lambda \in [0,1]$. *For* $f \in C[0,1]$ *and* $0 < \alpha < 2$, *the following statements are equivalent:*

(i) $|f(x) - B_k f(x)| = O\left(\left(\dfrac{\varphi^{2(1-\lambda)}(x)}{k} \right)^{\frac{\alpha}{2}} \right)$

$x \in [0,1], k = 1,2,\ldots,$

(ii) $\omega_2^{\varphi^{\lambda}}(f;\delta) = O(\delta^{\alpha})$, $\delta \to 0$.

We point out that for $\lambda = 0$ the second order Ditzian-Totik modulus reduces to the second order classical modulus. The natural question which arises is how to generalize the equivalence of Berens and Lorentz in order to include other possible values of *k* and *n* in the definition of the Schoenberg operator.

In Theorem 2 in [1] we proved that

$$(S_{n,k}(e_1 - x)^2)(x) \le 1. \frac{\min\left\{ 2x(1-x); \frac{k}{n} \right\}}{n+k-1},$$

for $n \ge 1, k \ge 1, \ x \in [0,1]$.

Now for $k=1$, $n \ge 1$-where $S_{n,1}$ reduces to the piecewise linear interpolant with equidistant interior knots, the last upper bound for the second moment is

$$(S_{n,1}(e_1 - x)^2)(x) \le \frac{1}{n^2}.$$

Otherwise, if now $n=1, k \ge 1$ -where $S_{1,k}$ - is the Bernstein polynomial of degree *k*, then we get

$$(S_{1,k}(e_1 - x)^2)(x) \le \frac{2x(1-x)}{k},$$

which corresponds to the wellknown representation of the second moment of the Bernstein operator. Therefore we may conclude, that the estimate for the second moment of the Schoenberg operator, given in Theorem 2 in [1] unifies the whole range of the parameters *n* and *k*. This estimate was applied by the second author in [10] to establish a Voronovskaja-type theorem for Schoenberg operator. As a consequence, we obtained.

Theorem C. (see Theorem 15 in [1]) *For all* $f \in C[0,1]$, $x \in [0,1]$, $h \in (0,1]$ *and* $n, k \ge 1$ *one has:*

$$|S_{n,k} f(x) - f(x)| \le$$

$$\le \left[1 + \frac{1}{2h^2} \cdot \frac{\min\left\{ 2x(1-x); \frac{k}{n} \right\}}{n+k-1} \right] . \omega_2(f,h).$$

In particular, one has

$$|S_{n,k} f(x) - f(x)| \le \frac{3}{2} . \omega_2\left(f; \sqrt{\frac{2x(1-x)}{n+k-1}} \right),$$

and

$$|S_{n,k} f(x) - f(x)| \le \frac{3}{2} . \omega_2\left(f; \sqrt{\frac{\min\left\{ 2x(1-x); \frac{k}{n} \right\}}{n+k-1}} \right).$$

For the so called "spline case" for *k* = 1, 2, 3 the following equivalence was established in [2], correspondent to the result in Theorem A:

Theorem D. *For* $k = 1,2,3$ *and* $f \in C[0,1]$ *the following is true:*

$$\| S_{n,k} f - f \|_\infty = O(n^{-\alpha}) \Leftrightarrow \omega_2(f,\delta) = O(\delta^\alpha),$$

$\delta \to 0, \alpha \in (0,2)$.

Our goal in this paper is to extend the result of Theorem D when $k \le n-1$ is fixed and $n \to \infty$. As a natural generalization of Theorems A and D, in [2] the following conjecture was made:

Conjecture. *For* $m \ge 1$ *let* (n_m) *and* (k_m) *be monotonically increasing sequences of natural numbers such that* $n_m \ge 2$, $k_m \ge 4$ *for all m and* $(n_m + k_m) \to \infty$, *when* $m \to \infty$.

Suppose that there exists a constant M independent of m and x such that

$$\left| \left(S_{n_m,k_m} f - f \right)(x) \right| \le$$

$$\le M . \left(\frac{\min\left\{ 2x(1-x); \frac{k_m}{n_m} \right\}}{n_m + k_m - 1} \right)^{\frac{\alpha}{2}}. \tag{1.3}$$

Then

$$\omega_2(f;h) \le c.M.h^\alpha, \ \frac{1}{2} > h > 0. \tag{1.4}$$

In the above conjecture, the case $n_m = 1$ is excluded, because then the result is implied by Theorem A for Bernstein operators. Moreover, the cases $k_m = 1,2,3$ were excluded since in those the statement follows from Theorem D. Throughout the paper with $\| \, . \, \|$, we denote the supremum norm over [0, 1]. Our main result states the following:

Theorem 1. *For* $f \in C[0,1]$, $\alpha \in (0,2)$ *and all* $n \ge k+1$, $k \ge 4$ *where n, k are natural numbers, k is fixed and* $n \to \infty$, *we suppose:*

$$\| f - S_{n,k} f \| \le M.n^{-\alpha} \tag{1.5}$$

and

$$\| f - S_{1,1} f \| \le M. \tag{1.6}$$

Then the following holds true:

$$\omega_2(f,h) \le CM.h^{-\alpha}, \ h \in \left(0, \frac{1}{2} \right], \tag{1.7}$$

where C is an absolute positive constant, independent of f, h.

Corollary 1. *For all* $n \ge k+1$, $k \ge 4$, *k - fixed and* $n \to \infty$ *if* $f \in C[0,1]$ *the following is true for all* $0 < \alpha < 2$:

$$\| S_{n,k} f - f \| = O(n^{-\alpha}) \Leftrightarrow \omega_2(f,\delta) = O(\delta^\alpha), \ \delta \to 0. \tag{1.8}$$

In Section 2, we prove some auxiliary lemmas. The proof of Theorem 1 and Corollary 1 is given in Section 3.

II. AUXILIARY LEMMAS

We start with the following Bernstein-type inequality, which was proved in [7] for $k = 1,2,3$:

Lemma 1. *For all,* $n \ge 1$ $k \ge 3$, *and all* $f \in C[0,1]$ *we have:*

$$\|S''_{n,k}f\| \le (3k(k-1)+4)n^2.\|f\|. \qquad (2.1)$$

Proof of Lemma 1: For the second derivative of the Schoenberg splines, we use the representation given by Marsden in [8]:

$$\left(S''_{n,k}f\right)(x) = \sum_{j=-k+2}^{n-1} a_j.k(k-1).N_{j,k-2}(x), \qquad (2.2)$$

$$a_j = \frac{\dfrac{f(\xi_{j,k}) - f(\xi_{j-1,k})}{x_{j+k}-x_j} - \dfrac{f(\xi_{j-1,k}) - f(\xi_{j-2,k})}{x_{j+k-1}-x_{j-1}}}{x_{j+k-1}-x_j}. \qquad (2.3)$$

We consider the following cases:

Case 1. $j = n-1$. ($j = -k+2$ is analogous). We verify that

$$\xi_{n-1,k} = 1, \qquad x_{j+k} - x_j = \frac{1}{n}$$

$$\xi_{n-2,k} = 1 - \frac{1}{nk}, \qquad x_{j+k-1} - x_{j-1} = \frac{2}{n}$$

$$\xi_{n-3,k} = 1 - \frac{3}{nk}, \qquad x_{j+k-1} - x_j = \frac{1}{n}.$$

$$|a_{n-1}| \le \left[\frac{2\|f\|}{\frac{1}{n}} + \frac{2\|f\|}{\frac{2}{n}}\right] n = 3n^2.\|f\|. \qquad (2.4)$$

Case 2. $n-2 \ge j > n-k$. ($-k+3 \le j < 1$ is analogous). We verify that

$$x_{j+k} = 1, \qquad x_{j+k} - x_j = 1 - \frac{j}{n}$$

$$x_{j+k-1} = 1, \qquad x_{j+k-1} - x_{j-1} = 1 - \frac{j-1}{n}$$

$$x_j = \frac{j}{n}, \qquad x_{j+k-1} - x_j = 1 - \frac{j}{n}.$$

$$|a_j| \le \left[\frac{2\|f\|.n}{n-j} + \frac{2\|f\|.n}{n-j+1}\right].\frac{n}{n-j} \le$$

$$\le \left[\frac{2\|f\|}{2} + \frac{2\|f\|}{3}\right].\frac{n^2}{2} \le \frac{5}{6}\|f\|n^2 < 3\|f\|n^2. \qquad (2.5)$$

Consequently, from (2.4) and (2.5) we obtain

$$\left[\sum_{j=-k+2}^{0} + \sum_{j=n-k+1}^{n-1}\right] |a_j| k(k-1).N_{j,k-2}(x) \le$$

$$\le 3k(k-1)n^2.\|f\|. \qquad (2.6)$$

Case 3. $1 \le j \le n-k$. (If $k \ge n$ this case is redundant). Simple calculations show that

$$|a_j| \le \frac{4\|f\|n^2}{k(k-1)}.$$

Therefore

$$\sum_{j=1}^{n-k} |a_j| k(k-1).N_{j,k-2}(x) \le 4\|f\|n^2. \qquad (2.7)$$

Estimates (2.7) and (2.6) complete the proof of Lemma 1.

The second Bernstein-type inequality is the following:

Lemma 2. *Let* $f \in C^2[0,1]$. *Then for* $n \ge k+1$ *the following holds true:*

$$\|S''_{n,k}f\| \le A\|f''\|, \qquad (2.8)$$

where we can take $A = \frac{5}{2}$.

Proof of Lemma 2: First we make the following observation. It is known that (see [9]) the Schoenberg operator $S_{n,k}f$ and its first derivative $S'_{n,k}f$ uniformly on [0, 1] approximate f and f' respectively. This is not the case for the second derivative. Nevertheless, inequality (2.8) shows that the norm of the second derivative of the Schoenberg operator is bounded by the norm of the second derivative of the approximated function f. The last fact is wellknown for the all derivatives of the classical Bernstein operator. We consider two cases:

Case 1. $1 \le j \le n-k$. Then

$$x_{j+k} - x_j = x_{j+k-1} - x_{j-1} = \frac{k}{n},$$

and

$$x_{j+k-1} - x_j = \frac{k-1}{n}.$$

Consequently, from (2.3) we get

$$|a_j| \le \frac{n^2}{k(k-1)}\omega^2(f, \frac{1}{n}).$$

Therefore

$$\left|\sum_{j=1}^{n-k} a_j k(k-1).N_{j,k-2}(x)\right| \le$$

$$\le \sum_{j=1}^{n-k} n^2\omega_2(f, \frac{1}{n}).N_{j,k-2}(x) \le n^2\omega_2(f, \frac{1}{n}) \le \|f''\|.$$

Case 2. $2-k \le j \le 0$. (The case $n-k+1 \le j \le n-1$ is similar). In this case, we have

$$x_{j+k} - x_j = \frac{s+1}{n}, \qquad x_{j+k-1} - x_{j-1} = \frac{s}{n},$$

where $s := j+k-1$ is a natural number, satisfying $1 \le s \le k-1$. Obviously

$$\xi_{j,k}-\xi_{j-1,k}=\frac{s+1}{kn}, \quad \xi_{j-1,k}-\xi_{j-2,k}=\frac{s}{kn}.$$

We denote $\xi_{j-2,k}=y_0$, $\frac{1}{kn}=h$. Then

$$\xi_{j-1,k}=y_0+sh,$$

$$\xi_{j,k}=(y_0+sh)+(s+1)h=y_0+(2s+1)h$$

and the numerator of (2.3) is equal to

$$\frac{f(y_0+(2s+1)h)-f(y_0+sh)}{(s+1)kh}-\frac{f(y_0+sh)-f(y_0)}{skh}=$$

$$=\frac{s[f(y_0+(2s+1)h)-f(y_0+sh)]}{s(s+1)kh}-$$

$$-\frac{(s+1)[f(y_0+sh)-f(y_0)]}{s(s+1)kh}=$$

$$=\frac{n}{s(s+1)}(s[f(y_0+(2s+1)h)-f(y_0+sh)]-$$

$$-(s+1)[f(y_0+sh)-f(y_0)]). \tag{2.9}$$

We consider the following subcases:

Case 2a. $y_0=\xi_{j-2,k}\ge h=\frac{1}{kn}$. This means that $j-2+k\ge 1,\ 3-k\le j\le 0$.

It is clear that $y_0-h\ge 0$, $y_0-h\in[0,1]$. From (2.9) we obtain

$$s[f(y_0+(2s+1)h)-f(y_0+sh)]-$$

$$-(s+1)[f(y_0+sh)-f(y_0)]=$$

$$=s(f(y_0+(2s+1)h)-2f(y_0+sh)+f(y_0-h))-$$

$$-(f(y_0+sh)-(s+1)f(y_0)+sf(y_0-h)). \tag{2.10}$$

The first summand in (2.10) is a second finite difference and therefore

$$|f(y_0+(2s+1)h)-2f(y_0+sh)+f(y_0-h)|\le\omega_2(f,(s+1)h)=$$

$$=\omega_2\left(f,\frac{s+1}{kn}\right)\le\left(\frac{s+1}{kn}\right)^2\|f''\|. \tag{2.11}$$

We represent the second summand in (2.10) as follows

$$f(y_0+sh)-(s+1)f(y_0)+sf(y_0-h)=$$

$$=f(y_0+sh)-2f(y_0+(s-1)h)+f(y_0+(s-2)h)+$$

$$+2(f(y_0+(s-1)h)-2f(y_0+(s-2)h)+f(y_0+(s-3)h))+$$

$$+3(f(y_0+(s-2)h)-2f(y_0+(s-3)h)+f(y_0+(s-4)h))+$$

$$+4(f(y_0+(s-3)h)-2f(y_0+(s-4)h)+f(y_0+(s-5)h))+$$

$$+...+s(f(y_0+h)-2f(y_0)+f(y_0-h)).$$

All summands in the last representation are second finite differences. Hence

$$|f(y_0+sh)-(s+1)f(y_0)+sf(y_0-h)|\le$$

$$\le(1+2+3+...+s)\omega_2(f,h)=\frac{s(s+1)}{2}\omega_2\left(f,\frac{1}{kn}\right)\le$$

$$\le\frac{s(s+1)}{2k^2n^2}\|f''\|. \tag{2.12}$$

Now the estimates (2.9), (2.10), (2.11) and (2.12) imply the following upper bound for the numerator of (2.3):

$$\frac{n}{s(s+1)}\left(\frac{s(s+1)^2}{k^2n^2}+\frac{s(s+1)}{2k^2n^2}\right)\|f''\|=$$

$$\frac{s}{k^2n}\left(1+\frac{3}{2s}\right)\|f''\|\le\frac{s}{k^2n}.\frac{5}{2}\|f''\|. \tag{2.13}$$

In case 2a we have

$$x_{j+k-1}-x_j=x_{j+k-1}=\frac{s}{n}.$$

The last observation, (2.3) and (2.13) imply

$$|a_j|\le\frac{5}{2k^2}\|f''\|. \tag{2.14}$$

Consequently

$$\left|\sum_{j=3-k}^{0}a_jk(k-1).N_{j,k-2}(x)\right|\le$$

$$\le\frac{5}{2}\|f''\|\sum_{j=3-k}^{0}N_{j,k-2}(x). \tag{2.15}$$

Case 2b. Let $y_0=\xi_{j-2,k}=0$. Therefore $j+k-2=0$, i.e. $j=2-k$. Simple calculations show

$$\xi_{j,k}-\xi_{j-1,k}=\frac{2}{kn},$$

$$\xi_{j-1,k}-\xi_{j-2,k}=\frac{1}{kn},$$

$$x_{j+k-1}-x_j=\frac{1}{n}.$$

If we denote $\frac{1}{kn}=h$ we observe that $\xi_{j-1,k}=h$, $\xi_{j,k}=3h$ and therefore the numerator in (2.3) can be represented as

$$\frac{n}{2}[f(3h)-f(h)-2f(h)+2f(0)]=$$

$$=\frac{n}{2}[(f(3h)-2f(2h)+f(h))+2(f(2h)-2f(h)+f(0))]\le$$

$$\le\frac{3}{2}n\omega_2(f,h)\le\frac{3}{2}nh^2\|f''\|=\frac{3}{2}n\frac{1}{k^2n^2}.\|f''\|.$$

Hence

$$\left|a_j\right| = \left|a_{2-k}\right| \le \frac{3}{2k^2}\|f''\|.$$

So we arrive at

$$\left|a_{2-k}k(k-1).N_{2-k,k}(x)\right| \le \frac{3}{2}\|f''\|.N_{2-k,k}(x). \quad (2.16)$$

The estimates (2.15), (2.16) and **case 1** complete the proof of Lemma 2.

The crucial step to establish the inverse results for many linear positive operators is the wellknown inverse lemma of Berens and Lorentz from 1972 given in the book of DeVore and Lorentz [4, Lemma 5.2, p.312] as:

Lemma 3. *If* $0<\alpha<2$*, if* ϕ *is an increasing, positive function on* $[0,a]$ *with* $\phi(0)=0$*, and if* $0<a\le 1$*, then the inequalities*

$$\phi(a) \le M_0 a^\alpha \quad (2.17)$$

and

$$\phi(x) \le M_0\left(y^\alpha + \frac{x^2}{y^2}\phi(y)\right),\ 0\le x\le y\le a, \quad (2.18)$$

imply for some $C=C(\alpha)>0$,

$$\phi(x) \le CM_0 x^\alpha,\ 0\le x\le a. \quad (2.19)$$

III. PROOF OF THEOREM 1

Proof of Theorem 1:

Our goal is to prove that the function $\phi(t)=\omega_2(f,t)$ satisfy the conditions (2.17) and (2.18). We recall that for $n=1$, $k\ge 1$ the Schoenberg operator coincides with the Bernstein operator, i.e. $S_{n,k}\equiv B_k$. It is known that

$$B_1 f(x) = f(0)(1-x) + f(1)x.$$

Therefore for $a=\frac{1}{2}$ we write

$$\omega_2(f,a) = \omega_2(f-B_1 f,a) \le 4\|f-B_1 f\| \le 4M, \quad (3.1)$$

using the fact that $f\in C[0,1]$ and (1.6). Without loss of generality we may suppose that $M=1$. Therefore the condition (2.17) will be fulfilled for $M_0\ge 2^{2+\alpha}$. To complete the proof of Theorem 1 it is enough to show

$$\omega_2(f,h) \le M_0.\left[\delta^\alpha + \frac{h^2}{\delta^2}.\omega_2(f,\delta)\right], 0\le h\le\delta\le a, \quad (3.2)$$

for sufficiently large M_0. For $0\le h\le\delta\le\frac{1}{5}$ we have

$$\omega_2(f,h) \le \omega_2(f-S_{n,k}f,h) + \omega_2(S_{n,k}f,h) \le$$

$$\le 4\|f-S_{n,k}f\| + h^2\|S''_{n,k}f\|. \quad (3.3)$$

We choose a natural number $n\ge k+1\ge 5$ satisfying

$$n\le\delta^{-1}\le 2n,\ \frac{1}{2n}\le\delta\le\frac{1}{n}.$$

From (1.5) it follows (with $M=1$)

$$\|f-S_{n,k}f\| \le n^{-\alpha} \le (2\delta)^\alpha = 2^\alpha.\delta^\alpha. \quad (3.4)$$

Due to the equivalence between the related $K-$ functional and second order modulus of continuity ω_2 (see Th. 2.4 in [4]-p.177), we may choose the function $f_\delta\in C^2[0,1]$ such that

$$\|f-f_\delta\| + \delta^2\|f''_\delta\| \le c_0\omega_2(f,\delta). \quad (3.5)$$

We proceed with the evaluation of the second term in (3.3). Consequently

$$S''_{n,k}f = S''_{n,k}(f-f_\delta) + S''_{n,k}(f_\delta).$$

Following Lemma 1 and (3.5) we get

$$\|S''_{n,k}(f-f_\delta)\| \le (3k(k-1)+4)n^2\|f-f_\delta\| \le$$

$$\le c_1 n^2\omega_2(f,\delta), \quad (3.6)$$

with $c_1=(3k(k-1)+4)c_0$. We apply Lemma 2 to obtain

$$\|S''_{n,k}f\| \le \|S''_{n,k}(f-f_\delta)\| + \|S''_{n,k}(f_\delta)\| \le$$

$$\le c_1 n^2\omega_2(f,\delta) + \frac{5}{2}\|f''_\delta\| \le$$

$$\le c_1 n^2\omega_2(f,\delta) + \frac{5}{2}\delta^{-2}\omega_2(f,\delta) \le$$

$$\le c_2.\delta^{-2}\omega_2(f,\delta), \quad (3.7)$$

with $c_2=c_1+\frac{5}{2}$. The estimates (3.7), (3.4) and (3.3) imply (3.2) for $0\le h\le\delta\le\frac{1}{5}$. To extend the last statement for $0\le h_1\le\delta_1\le\frac{1}{2}$, we observe that for $h_1:=\frac{5}{2}h$, $\delta_1:=\frac{5}{2}\delta$ with $0\le h\le\delta\le\frac{1}{5}$ it follows $0\le h_1\le\delta_1\le\frac{1}{2}$. Then from the properties of the second order moduli of continuity we obtain

$$\omega_2(f,h_1) = \omega_2\left(f,\frac{5}{2}h\right) \le \left(\frac{5}{2}+1\right)^2.\omega_2(f,h) \le$$

$$\le \left(\frac{5}{2}+1\right)^2.M_0.\left[\delta^\alpha + \frac{h^2}{\delta^2}.\omega_2(f,\delta)\right] \le$$

$$\le \left(\frac{5}{2}+1\right)^2.M_0.\left[\left(\frac{2}{5}\right)^\alpha.\delta_1^\alpha + \frac{h_1^2}{\delta_1^2}.\omega_2(f,\delta_1)\right].$$

The both conditions in Berens-Lorentz Lemma 3 are satisfied. Thus, the proof of Theorem 1 is completed.

Remark 1. The proof of Corollary 1 follows immediately from Theorem 1 and Theorem C.

Remark 2. We point out that the estimate (1.4) is proved in Theorem 1 by assumption (1.5) - weaker than (1.3). It is also possible to remove the condition (1.6), but in this case the constant C in (1.7) will depend on $\|f\|$.

Remark 3. For the polynomial case of Schoenberg operator, i.e $n \le k$, n-fixed and $k \to \infty$, we believe that the proper constructive characteristic instead of $\omega_2(f,\delta)$ should be $\omega_2^{\varphi}(f,\delta)$ -the second order Ditzian-Totik modulus of smoothness.

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